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Extension theorems with the range space not necessarily Dedekind complete

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Abstract. We show that every positive linear operator from a majorizing subspace of a separable Fréchet lattice into a Hausdorff locally solid Riesz space with the Fatou property and the σ-interpolation property can be extended. We shall also characterize the extreme points of the convex set of all positive linear extensions of a positive linear operator defined on a vector subspace when the range space is not assumed to be Dedekind complete.

Keywords: locally solid Riesz space, positive linear operator, Hahn-Banach type theorems, σ-interpolation property

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1 Introduction

In the classical Hahn-Banach-Kantorovich theorem, the range space is assumed to be Dedekind complete. This assumption can be relaxed by using the ideas of Y. A. Abramovich and A. W. Wickstead (see [1]) about the interplay between the separability of the domain space and the σ-interpolation property of the range space.

1 Theorem ([1], Theorem 3.5)**.** *Let E and F be Banach lattices such that E is separable and F has the σ-interpolation property, and let $P : E \to F$ be a continuous sublinear operator. If G is a vector subspace of E and $T : G \to F$ is a continuous linear operator satisfying $T(x) \leq P(x)$ for all $x \in G$, then there exists a continuous linear extension $\hat{T}$ of T to all of E also satisfying $\hat{T}(x) \leq P(x)$ for all $x \in E$.*

In [5], using the above result N. Dănet confirmed the following conjecture posed by Wickstead in [14]: *the space of all continuous regular operators $\mathcal{L}^r(E,F)$ between the Banach lattices E and F has the Riesz decomposition property when E is separable and F has the σ-interpolation property.* For the study of the order structure of the space of regular operators, N. Dănet extended

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Theorem 1 in a more general setting and obtained the following version of the Hahn-Banach extension theorem.

2 Theorem ([6]). *Let (X, τ_1) be a metrizable topological vector space such that X is τ_1-separable, and let (F, τ_2) be an ordered topological vector space with the positive cone F^+ τ_2-closed and which has the strong σ-interpolation property. Let $P : X \to F$ be a continuous sublinear operator. If G is a vector subspace of X and $T : G \to F$ is a continuous linear operator which satisfies $T(x) \le P(x)$ for all $x \in G$, then there exists a continuous linear extension $\hat{T}$ of T to all of X such that $\hat{T}(x) \le P(x)$ for all $x \in X$.*

Recently, R. M. Dănet and N. C. Wong have used the ideas of Abramovich and Wickstead to prove new results regarding the extension of the positive operators. We list one here and refer to [7], [8] and [9] for more results.

3 Theorem ([9], Theorem 5). *Let E and F be two Banach lattices such that E is separable and F has the σ-interpolation property. Let $P : E \to F^+$ be a continuous sublinear operator. Suppose that G is a vector sublattice of E and $T : G \to F$ is a positive linear operator such that $T(a) \le P(a)$ for all a in G. If P is monotone on E^+, then there exists a positive linear operator $S : E \to F$ extending T such that $S(x) \le P(x^+)$ for all x in E. In case P is a lattice seminorm, we obtain $S(x) \le P(x)$, for all x in E.*

The argument in the proof of Theorem 3 indeed works also for the case when E is a metrizable locally solid Riesz space such that E is separable, and F is a Hausdorff locally solid Riesz space with the σ-interpolation property.

In this paper, we present some new extension theorems of this sort. In particular, we shall show that every positive linear operator from a majorizing subspace of a separable Fréchet lattice into a Hausdorff locally solid Riesz space with the Fatou property and the σ-interpolation property can be extended. We shall also characterize the extreme points of the convex set of all positive linear extensions of a positive linear operator defined on a vector subspace when the range space is not assumed to be Dedekind complete.

2 Preliminaries

In this paper, all spaces are over the reals. Recall that an ordered vector space E is said to have the *σ-interpolation property* (or the *Cantor property*) if for every increasing sequence (x_n) and every decreasing sequence (y_n) in E with $x_n \le y_n$, $\forall\ n \in \mathbb{N}$, there is an element z in E such that $x_n \le z \le y_n$, $\forall\ n \in \mathbb{N}$. E is said to have the *strong σ-interpolation property* if for every pair of sequences (x_n) and (y_m) in E with $x_n \le y_m$, $\forall\ n, m \in \mathbb{N}$, there is a z in E such that $x_n \le z \le y_m$, $\forall\ n, m \in \mathbb{N}$. In case E is a Riesz space, i.e. vector lattice, these two notions coincide. If Ω is a completely regular Hausdorff space, then the space $C(\Omega)$ has the σ-interpolation property if and only if any pair of disjoint open F_σ subsets of Ω have disjoint closures (see [13]). Huijsmans and Pagter have shown that an Archimedean Riesz space E has the σ-interpolation property if and only if E is uniformly complete and $E = \{ x^+ \}^\perp + \{ x^- \}^\perp$ for all $x \in E$ (see [10]).

Let E be an ordered vector space. Recall that a subset A of E is called to be *full* if $a_1,\ a_2 \in A$ and $a_1 \leq a_2$ imply that the order interval $[a_1, a_2] \subseteq A$. A linear topology τ on E is called *locally full* if there exists a neighbourhood basis of the origin consisting of full sets. An ordered vector space endowed with a locally full topology is called an *ordered topological vector space*. If the locally full topology τ on E is such that the positive cone E^+ is closed, then τ is a Hausdorff topology and E is Archimedean ([4, pp. 159–160]).

Let F be a Riesz space. A subset S of F is said to be *solid* if $|y| \leq |x|$ and $x \in S$ imply $y \in S$. A linear topology τ on F is called *locally solid* if there exists a neighbourhood basis of the origin consisting of solid sets. A Riesz space endowed with a locally solid topology is called a *locally solid Riesz space* (see [2]) or a *topological Riesz space* (see [4]). A metrizable locally solid topology τ on a Riesz space F is said to be *a Fréchet topology* if (F, τ) is τ-complete. A *Fréchet lattice* is a Riesz space equipped with a Fréchet topology. Recall that a subset S of a Riesz space is called *order closed* if it follows from $x_\alpha \xrightarrow{(o)} x$ and $\{ x_\alpha \} \subseteq S$ that $x \in S$. We say that a locally solid Riesz space (F, τ) satisfies the *Fatou property* (or that τ is a *Fatou topology*) if τ has a neighbourhood basis of the origin consisting of solid and order closed sets.

Let E and F be Riesz spaces. By a *sublinear operator* we mean an operator that is subadditive and positively homogeneous. An operator $P : E \to F^+$ is said to be a (*vectorial*) *seminorm* if P has the properties of a seminorm like in the scalar case. Also, a seminorm $P : E \to F^+$ is called a *lattice seminorm* if $P(x_1) \leq P(x_2)$ in F^+ whenever $|x_1| \leq |x_2|$ in E.

For the unexplained terminology see [2], [3] or [4].

3 Extension theorems for positive operators

We begin with a dominated extension type result.

4 Theorem. *Let E be a separable metrizable locally solid Riesz space and F be a Hausdorff locally solid Riesz space with the σ-interpolation property. Let $P : E \to F^+$ be a continuous lattice seminorm. Suppose that G is a Riesz subspace of E, and $T_1, T_2, \ldots, T_n$ are positive linear operators from G into F such that $\sum_{i=1}^n T_i(u) \leq P(u)$, $\forall\ u \in G$. Then each T_i can be extended to a* *positive linear operator* $S_i : E \to F$ *such that* $\sum_{i=1}^{n} S_i(x) \leq P(x)$, $\forall\, x \in E$.

Proof. It is enough to establish the result for $n = 2$. Let $T = T_1 + T_2$. By Theorem 3 (and the remarks that follow) there exists a positive linear operator $S : E \to F$ extending T such that $S(x) \leq P(x)$ for all x in E. Note that S is continuous, since S is dominated by P. Define a continuous lattice seminorm $P_1(x) = S(|x|), x \in E$. Then $T_1(u) \leq T_1(|u|) \leq T(|u|) = S(|u|) = P_1(u)$ for all u in G. By Theorem 3 again we obtain a continuous positive extension S_1 of T_1 such that $S_1(x) \leq P_1(x)$ for all $x \in E$. Let $S_2 = S - S_1$. Clearly S_2 is a positive linear operator extending T_2. Thus S_1, S_2 are the desired positive linear operators. $\boxed{QED}$

The following result is a generalization of [9, Theorem 10] which is itself inspired by [4, p. 210].

5 Theorem. *Let E be an ordered vector space equipped with a metrizable locally convex topology τ_1 such that E is τ_1-separable, and let (F, τ_2) be an ordered topological vector space with the positive cone F^+ τ_2-closed and having the strong σ-interpolation property. Let T be a positive linear operator defined on a vector subspace G of E with values in F. If there exists a neighbourhood U of 0 in E such that the set*

$$S = \{\, T(x) : x \in G,\, x \leq u\, for\, some\, u \in U \,\}$$

is bounded from above, then T can be extended on E to a continuous positive linear operator.

Proof. We can assume U convex and balanced. Put $M = U - E^+$. Clearly, $S = \{\, T(x) : x \in G \bigcap M \,\}$. Let S be bounded above by $y_0 \in F^+$. Note that M is a convex neighbourhood of zero in E, and let P_M be the continuous sublinear Minkowski functional of M. Define the continuous sublinear operator $P : E \to F^+$ by $P(x) = P_M(x)y_0$. For every x in G and every $\varepsilon > 0$, we have $x \in (P_M(x) + \varepsilon)M$, thus $T(x) \leq P(x)$. Here we use the hypothesis that F^+ is closed. Now by Theorem 2 there exists a continuous linear operator $\hat{T}$ extending T on E such that$\hat{T}(x) \leq P(x)$ for all $x \in E$.

Let us prove that $\hat{T}$ is positive. To this end, let $x \in E_+$ and $n \in \mathbb{N}$. Since $-nx \in M$, we have $P(-nx) = P_M(-nx)y_0 \leq y_0$. Therefore, $\hat{T}(x) \geq -\frac{1}{n}y_0$ for $n \in \mathbb{N}$, which implies that $\hat{T}(x) \geq 0$. $\boxed{QED}$

Based on Theorem 5 just proved, we give a result whose proof is the same as that of [4, p. 211, Proposition 2], thus omitted.

6 Corollary. *Let E be an ordered vector space equipped with a metrizable locally convex topology τ_1 such that E is τ_1-separable, and let F be an ordered vector space such that $F = F^+ - F^+$ and F has the strong σ-interpolation property. Suppose that τ_2 is a locally convex and locally full topology on F such that* *the positive cone F^+ is τ_2-closed. Then there is a nonzero positive, continuous linear operator of E into F if and only if E^+ is not τ_1-dense in E.*

Let us say that a vector subspace G of an ordered vector space E is a majorizing subspace whenever for each $x \in E$ there exists some $u \in G$ with $x \leq u$ (or, equivalently, whenever for each $x \in E$ there exists some $u \in G$ with $u \leq x$). Our next result is inspired by Theorem 2 of [8].

7 Theorem. *Let E be a separable Fréchet lattice and F a Hausdorff locally solid Riesz space with the σ-interpolation property and the Fatou property. Let G be a majorizing subspace of E. If $T : G \to F$ is a positive linear operator, then T has a positive linear extension to all of E.*

Proof. We claim that T is continuous. Indeed, let τ be the Fatou topology on F. Then there exists a unique Fatou topology τ^δ on the Dedekind completion F^δ of F that induces τ on F ([2, Theorem 11.10]). We may consider T as a mapping of G into F^δ. Then $T : G \to (F^\delta, \tau^\delta)$ has a continuous positive extension to E ([3, Theorem 2.8]; [2, Theorem 16.6]). Therefore, $T : G \to F$ is continuous.

Now let $x_0 \in E \setminus G$ and G_1 the linear span of $G \bigcup \{ x_0 \}$. We shall extend T to a positive operator on G_1. Let $A = \{ u \in G : u \leq x_0 \}$ and $B = \{ v \in G : x_0 \leq v \}$. Since G majorizes E, both A and B are nonempty. For every $u \in A$ and every $v \in B$ we have $u \leq x_0 \leq v$, from which we obtain

$$T(u) \leq T(v), \quad \forall\, u \in A, v \in B. \tag{1}$$

Since E is separable and the topology on E is metrizable, the sets A, B are also separable. Let A_1, B_1 be dense countable subsets of A and B respectively. In particular, the inequality (1) holds for all $u \in A_1$ and all $v \in B_1$. Since F has the σ-interpolation property, there exists an element y_0 in F such that

$$T(u) \leq y_0 \leq T(v), \quad \forall\, u \in A_1, v \in B_1.$$

Because T is continuous and F is Hausdorff locally solid Riesz space, the above inequality is true for all $u \in A$ and $v \in B$. Therefore

$$T(u) \leq y_0 \leq T(v), \quad \forall\, u \in A, \quad v \in B. \tag{2}$$

We define the operator $T_1 : G_1 \to F$ by putting

$$T_1(a + \lambda x_0) = T(a) + \lambda y_0, \quad \forall a \in G, \quad \lambda \in \mathbb{R}.$$

Clearly, T_1 extends T. We claim that T_1 is positive. To this end, Let $a \in G$ and $\lambda \neq 0$ such that $a + \lambda x_0 \geq 0$. If $\lambda > 0$, then $x_0 \geq -\frac{1}{\lambda} a$, which implies $T(-\frac{1}{\lambda} a) \leq y_0$. Hence $T_1(a + \lambda x_0) = T(a) + \lambda y_0 \geq 0$. The same is true for the case $\lambda < 0$. Finally, note that G_1 is again a majorizing subspace and T_1 is also continuous. The proof is finished by using Zorn's lemma. $\boxed{QED}$

4 Characterization of extreme extensions

Let E and F be Riesz spaces and let G be a vector subspace of E. Consider a positive operator $T : G \to F$. By $\mathcal{E}(T)$ we shall denote the collection of all positive linear extensions of T to all of E. Let $\operatorname{extr}\mathcal{E}(T)$ denote the set of all extreme points of the convex set $\mathcal{E}(T)$. In many instances, $\operatorname{extr}\mathcal{E}(T)$ may happen to be empty even if $\mathcal{E}(T) \neq \emptyset$. In case F is Dedekind complete, Z. Lipecki, D. Plachky and W. Thomson have shown the following : $S \in \operatorname{extr}\mathcal{E}(T)$ if and only if $\inf\{\, S(|x-u|) : u \in G\,\} = 0$ for each $x \in E$ ([11]; [12, Theorem 2]). The Dedekind completeness of F is indispensable in their investigation of extreme extensions. Next, we shall present a result about extreme extensions, where the range space is only assumed to have the σ-interpolation property admittedly weaker than Dedekind completeness.

8 Theorem. *Let E be a separable Fréchet lattice and F a Hausdorff locally solid Riesz space with the σ-interpolation property. Let G be a vector subspace of E and $T : G \to F$ a positive linear operator. Then for an operator $S \in \mathcal{E}(T)$ the following statements are equivalent:*

(i) $S \in \operatorname{extr}\mathcal{E}(T)$.

(ii) For any continuous seminorm $P : E \to F^+$ satisfying

$$P(x) \leq S(|\,x-u|\,), \quad \forall\, x \in E, u \in G \tag{3}$$

we have $P = 0$ on E.

Proof. (i)$\Longrightarrow$(ii) Let $P : E \to F^+$ be a continuous seminorm satisfying the inequality (3). Clearly, $0 \leq P(x) = P(-x) \leq S(|\,x|\,)$ for all $x \in E$, and also $P(u) = 0$ for every $u \in G$. Now, we claim that $P(x) = 0$ for all $x \in E$. To this end, we can assume by way of contradiction that there exists some element $x_0 \in E$ such that $P(x_0) > 0$ in F. Let E_0 be the set $\{\,\lambda x_0 : \lambda \in \mathbb{R}\,\}$. Define the operator $R_0 : E_0 \to F$ by $R_0(\lambda x_0) = \lambda P(x_0)$. Obviously, R_0 is a continuous linear operator, and $R_0(\lambda x_0) \leq P(\lambda x_0)$. By Theorem 2, R_0 has a continuous linear extension to all of E denoted by R such that $R(x) \leq P(x)$ for all $x \in E$. Clearly, $R \neq 0$, and $|R(x)| \leq P(x)$ holds for all $x \in E$. Therefore we can easily see that $R = 0$ on G. Since for each $x \geq 0$ in E we have

$$R(x) \leq P(x) \leq S(x)$$

and

$$-R(x) = R(-x) \leq P(-x) = P(x) \leq S(x),$$

it easily follows that $S - R \geq 0$ and $S + R \geq 0$. Hence, $S - R$ and $S + R$ are all elements of $\mathcal{E}(T)$. Note that $S - R \neq S$ and $S + R \neq S$. Now the equality

$S = \frac{1}{2}(S-R)+\frac{1}{2}(S+R)$ implies that S is not an extreme point of $\mathcal{E}(T)$, which leads to a contradiction. Therefore, $P = 0$ on E.

(ii)$\Longrightarrow$(i) Let S satisfy the requirements in (ii) and let $S = \lambda S_1 + (1-\lambda)S_2$ with $S_1, S_2 \in \mathcal{E}(T)$ and $0 < \lambda < 1$. Then for each $x, y \in E$ we have

$$|S_1(x) - S_1(y)| \le S_1(|x-y|\,) = \left(\frac{1}{\lambda}S - \frac{1-\lambda}{\lambda}S_2\right)(|x-y|\,) \le \frac{1}{\lambda}S(|x-y|\,).$$

Next, for each $x \in E$ and each $u \in G$, it follows from $S(u) = S_1(u) = S_2(u)$ that

$$|S(x) - S_1(x)| \le |S(x) - S(u)| + |S_1(u) - S_1(x)| \le \left(1 + \frac{1}{\lambda}\right) S(|x-u|\,)$$

Now Put $P(x) = \frac{\lambda}{1+\lambda}|S(x) - S_1(x)\,|$. Since the positive operators S, S_1 are both continuous ([2, Theorem 16.6]), $P(x)$ is a continuous seminorm which satisfies the inequality (3). By the hypothesis, we have $P(x) = 0$, which implies that $S = S_1$ on E. Therefore, S is an extreme point of $\mathcal{E}(T)$. Now the proof is finished. $\boxed{QED}$

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